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Multifractal Magnetization on Hierarchical Lattices

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A new approach is applied to show that the local magnetization of the ferromagnetic Ising model on hierarchical lattices has a multifractal structure at the critical point. The $f(\alpha)$ function characterizing its multifractality is presented and discussed for the diamond hierarchical lattice. Distinct exact critical exponents for the average magnetization and for the local magnetization of the deepest sites are found. The average magnetization (as function of the temperature) is also calculated. The critical exponent of the susceptibility is estimated using finite-size scale arguments.

KEY WORDS: Multifractality; Ising model; hierarchical lattices; magnetization.

The study of spin models on hierarchical lattices became relevant as a real-space renormalization-group approach after the Migdal-Kadanoff RG equations[(1)] were proved to be exact on the diamond hierarchical lattice (DHL hereafter).[(2)] Several properties of the Ising model on these lattices were established. For instance, the thermodynamic limit of the free energy has been proved to be well defined.[(3,4)] The total spontaneous magnetization has been analytically obtained[(4)] and the total susceptibility has been found to be infinite for $T > T_c$,[(4,6)] and finite for $T < T_c$.[(4,7)] Now, we show in this communication that the local magnetization of the ferromagnetic Ising model on hierarchical lattices has a multifractal structure. This is found through a new approach that generates an exact recurrence equation relating the local magnetization of a given site to the ones of previous distinct hierarchical levels. In addition, our approach allows one to obtain exact thermodynamic functions (such as the magnetization) and critical

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exponents. As far as we know, spin models with intrinsic multifractal local magnetization have not been reported in the literature. Fractal local magnetization has been found in discrete and continuous spin chains in a binary random field, but the fractal features resulted from the discreteness of the random field.[8] In the present paper, the multifractality is an intrinsic property induced by the topology of the lattice. Actually the hierarchical lattices are themselves topological fractal objects in the sense that they are scale invariant without being translation invariant. Furthermore, the coordination number varies exponentially from finite to infinite figures. Therefore, fractal features should appear on the thermodynamic and critical properties of spin models on these lattices. For instance, the zeros of the partition function of the Ising model on the DHL have been shown to have a fractal structure.[9]

We consider here the ferromagnetic Ising model on the simplest hierarchical lattice, namely the diamond hierarchical lattice. Starting with a bond linking the two roots A and B (so-called terminals by Tsallis[5] and surface sites by Kaufman and Griffiths[6]), the DHL is constructed by replacing this bond by the DHL's basic cell (see Fig. 1a). The new bonds are also replaced by the basic cell and so successively, giving the different N levels of the DHL (see Fig. 1b). The reduced Hamiltonian of the spin-1/2 ferromagnetic Ising model with zero field in an N-level DHL is (the zero-level DHL is the initial bond)

$$-\beta H_N = K_N \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{1}$$

where K_N is the reduced coupling constant of the exchange interaction between all pairs $\langle ij \rangle$ of nearest neighbor spins of the N-level DHL, and the σ's are the spin variables ($\sigma = \pm 1$). In order to analyze the structure of the local magnetization, it is sufficient to consider only the sites of one of the shortest paths joining the two roots of the DHL. All the shortest paths are equivalent, since each one contains (in a symmetrically arranged way with respect to the middle point) all kinds of sites of the DHL with distinct coordination numbers and depths with respect to the roots. We can identify the sites on this path by a pair (s, l), where l is the level ($l = 0, 1, 2,..., N$) and s is the position ($s = 1, 3, 5, 7,..., 2^l - 1$) of the site within the lth level with respect to one root (see Fig. 1c). If $l = N$, then s is the chemical distance from the considered site (s, N) to one of the roots, say the root A. For example, in Fig. 1b, the minimum number of steps between the site (7, 3) and the root A is 7. If $l < N$, then the chemical distance s must be calculated in the l-level DHL. For example, in Fig. 1b the chemical distance of the site (1, 1) and A is 1 (see the bottom of Fig. 1a).

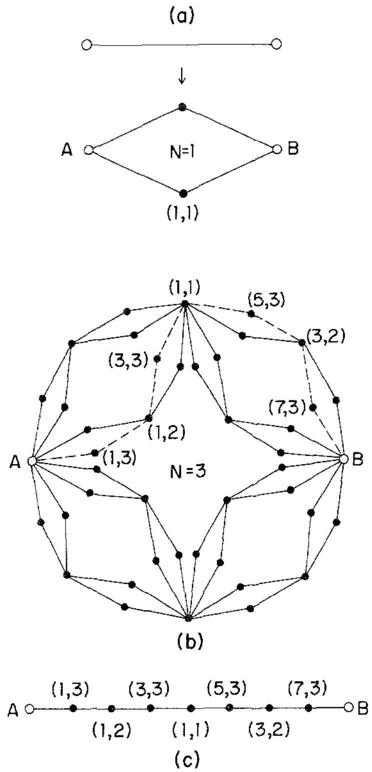

Fig. 1. Diamond hierarchical lattice: (a) One bond replaced by the basic cell. (b) The DHL up to three levels. The open circles are the roots. The broken line indicates an arbitrary shortest path joining the roots. (c) The sequence of sites (s, l) appearing in a shortest path between the roots A and B for the three-level DHL.

We found that the local magnetization $m_{s,l}$ of a given site s of the lth level ($l = 1,..., N$) is related to the local magnetizations of its nearest neighbor sites at the lth level, one belonging to the $(l-1)$th level and the other to the jth level ($j = 0, 1, 2,..., l-2$), by[3]

$$m_{s,l} = A_l(m_{s',l-1} + m_{s'',j}) \tag{2}$$

[3] Equation (2) can be obtained straightforwardly by evaluating first the local magnetization of a site of the last hierarchical level $l = N$ and of its nearest-neighbor sites as functions of the coupling constant K_N, the effective fields, and the effective interaction acting upon these neighbors. The effective couplings can be obtained by tracing over the spins different from these three. Equation (2) is obtained, then, by eliminating the effective fields and the effective interaction parameter as functions of the local magnetizations.

with $s' = (s \pm 1)/2$, $s'' = (s \mp 1)/2^{l-j}$, and

$$A_l = t_l/(1 + t_l^2) \tag{3}$$

where $t_l = \tanh K_l$. For temperatures below the ferromagnetic critical temperature T_c, the local magnetization profile of the DHL can be obtained from Eqs. (2) and (3), and with the help of the renormalization equation for the coupling constants of the DHL, given by

$$t_{l-1} = 2t_l^2/(1 + t_l^4) \tag{4}$$

This can be done numerically starting with $m_{1,1} = 2A_1$, which corresponds to having the spins at the roots with fixed values $\sigma = 1$.[4] However, at the critical temperature one has $A_l \Rightarrow A_c = t_c/(1 + t_c^2) \cong 0.419643$, where t_c is the unstable fixed-point solution of Eq. (4), namely

[4] Actually, to solve Eqs. (2)–(4), it is necessary only to fix *one* spin in order to break the symmetry. However, this leads to an asymmetric magnetization profile.

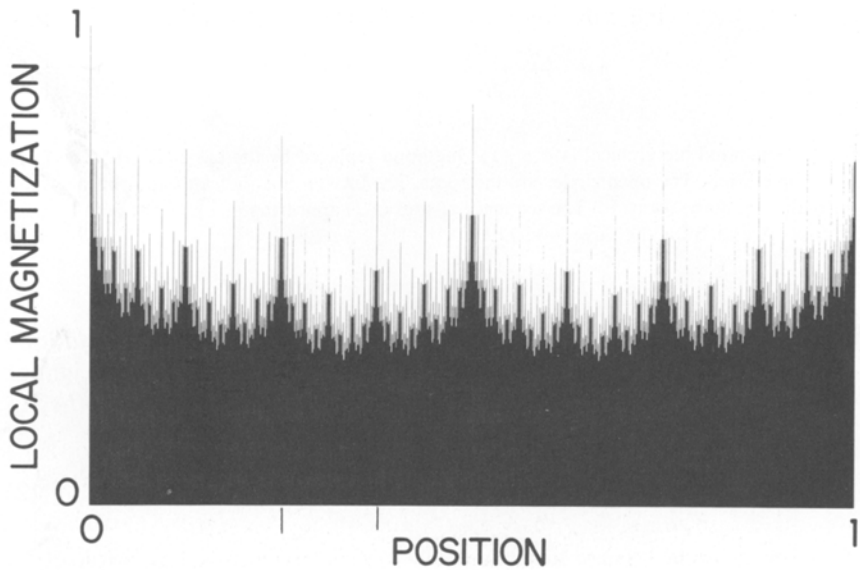

Fig. 2. Local magnetization (at the critical point) against spin position (on an arbitrary scale) within a shortest path joining the two roots of a DHL with $N = 10$ levels. Note: the average magnetization of this profile does vanish at the $N \Rightarrow \infty$ thermodynamic limit, which is not evident from this $N = 10$ plot.

$t_c = (a_c - 2/a_c - 1)/3 = 0.543689...$, with $a_c = (3\sqrt{33} + 17)^{1/3}$. Therefore the magnetization profile at T_c is straightforwardly given by Eq. (2) with A_c replacing A_l. In Fig. 2 we show the local magnetization profile at T_c of the DHL for $N = 10$ levels. We remark that the average magnetization of this magnetization profile at the critical temperature vanishes in the $N \Rightarrow \infty$ thermodynamic limit. It is easy to see that Fig. 2 has no smallest scale. For instance, in Fig. 3 we display the magnification of Fig. 2 between two deep sites, showing thus its fractal nature. One can show that this magnetization profile in the infinite-level limit (thermodynamic limit) is an everywhere discontinuous function. For every value of m, there is an $mA_c/(1 - A_c) \cong 0.72m$ discontinuity both to the left and to the right limits. We also remark the similarity of the form of Fig. 2 with that of the celebrated Weierstrass function,[10] which is actually continuous everywhere but differentiable nowhere. Moreover, it can be shown that the local magnetization distribution has a multifractal structure. In fact the $f(\alpha)$ function, which describes how densely the singularities of a measure are distributed,[11] can be obtained assuming for our measure the normalized local magnetization. This can be done numerically by evaluating the $N \Rightarrow \infty$ limit behavior of the reduced double logarithmic plot of the measure distribution calculated for a finite N with a box width (2^{-N}), where $\ln(2^{-N})$ is the reducing factor.[12] Figure 4 displays the $f(\alpha)$ function

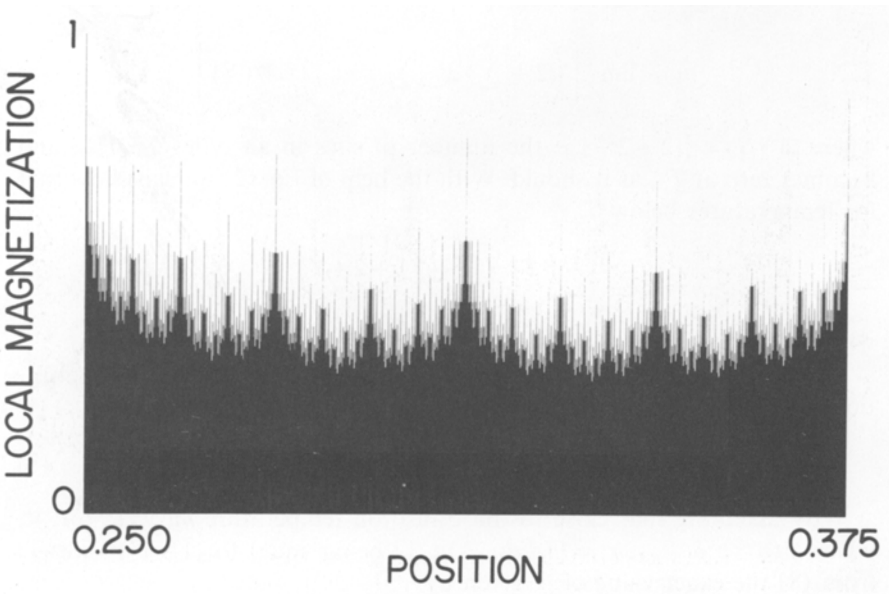

Fig. 3. Magnification of Fig. 1 between 0.250 and 0.375 on a renormalized scale.

corresponding to the measure spectrum of the local magnetization of a DHL for $N=29$ levels which corresponds, in practice, to the thermodynamic limit for temperatures close to the critical point. Despite the slow convergence, Fig. 4 contains most features of the $f(\alpha)$ function, say: a concave curve with a single maximum at the Hausdorff dimension of the support (which is one in our case), and an infinite slope at the points in which $f(\alpha)=0$. In this figure, we also show what we expect to look like the true (dashed line) $f(\alpha)$ function ($N \Rightarrow \infty$ limit). The $\alpha_{\min}$ and the $\alpha_{\max}$, which reflect, respectively, how the measures of the most concentrated and the most rarefied intervals scale with the box width, are calculated exactly. Actually, we get

$$\alpha_{\min} = -\ln t_c/\ln 2 \cong 0.8791... \tag{5}$$

and

$$\alpha_{\max} = 2 - (\ln\{(1-t_c)[1+(1+4t_c^{-1}+4t_c)^{1/2}]\}/\ln 2) \cong 1.0460...$$

$\alpha_{\max}$ is in extremely good agreement with the numerical endpoint of the graph for $N=29$ levels. The $\alpha_{\min}$ as well as the value $f_{\max}(\alpha)=1$ value has slow convergence. In the inset of Fig. 4 we show the values of $f_{\max}(\alpha)$ as a function of N^{-1}, which indicates this convergence.

The average magnetization (per site) of the entire lattice is defined by

$$m = \lim_{N \Rightarrow \infty} \left[\left(2 + \sum_{l=1}^{N} 2^l \sum_{s=1,3...}^{2^l-1} m_{s,l} \right) \Big/ \mathcal{N}(N) \right] \tag{6}$$

where $\mathcal{N}(N) = \frac{2}{3}(2+2^{2N})$ is the number of sites in an N-level DHL, and becomes zero at T_c, as it should. With the help of Eq. (2) we can show that for temperatures below T_c

$$m(T) = \prod_{i=1}^{\infty} \frac{1}{2}(1+2A_i) \tag{7}$$

which recovers a previous result.[4]

From Eqs. (2) and (6) one is able to show that the average magnetization also follows a recursion equation which at T_c reads

$$m(N) = \tfrac{1}{4}(1+2t_c^{-1})\, m(N-1) - \tfrac{1}{8}(t_c^{-1})\, m(N-2) \tag{8}$$

By assuming that close to the transition temperature $m(N) \propto (\delta t_N)^\beta$, where $\delta t_N \cong r_c \delta t_{N+1}$, $r_c = [(d/dt_{n+1})\, t_n]_{t_c}$ being equal to $(1+t_c^2)^2$, we get from (8) the exact value of β, given by

$$\beta = \ln(2t_c)/2\ln(1+t_c^2) \cong 0.161734374... \tag{9}$$

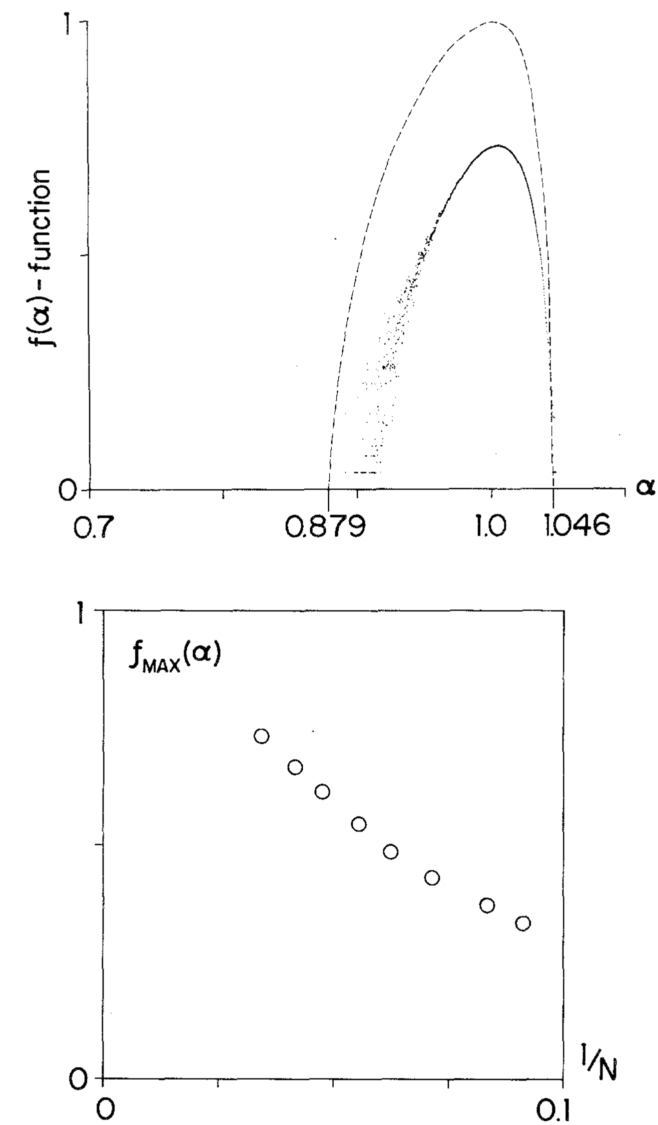

Fig. 4. The $f(\alpha)$ function of the DHL magnetization profile for $N = 29$ levels. The dashed curve represents the expected $N \Rightarrow \infty$ limit. The inset gives the maximum values of $f(\alpha)$ as a function of N^{-1}, showing the slow convergence to one as $N \Rightarrow \infty$.

β is the critical exponent associated with the average magnetization of the entire lattice, which has been already obtained by other approaches.[4,13] From Eq. (2), using similar arguments, we also calculate exactly the critical exponent β' corresponding to the measure of the most rarefied box width of Fig. 2, say, the normalized local magnetization of the deepest sites of the DHL. This leads to

$$\beta' = \tfrac{1}{2}(1 - \ln\{t_c[1 + (1 + 4t_c^{-1} + 4t_c)^{1/2}]/2\}/\ln(1 + t_c^2)) \cong 0.22335...$$

One can also calculate from (6) the total magnetization $M_N(T)$, showing that for an Nth-level DHL at T_c, $M_N(T_c) \propto (2/t_c)^N$. Let us assume the conjecture made for two- and three-dimensional Ising models,[14] namely, that at T_c the magnetization shows a power-law behavior with respect to the linear size (L) of the system, that is, $M(L) \propto L^D$ under scale transformation. We thus obtain for the DHL that $M_N(T_c) \propto (2^N)^D$ with $D = \ln(2/t_c)/\ln(2) = 1.87914...$, where D corresponds to the fractal dimension of the set of sites of the DHL with arbitrarily small but finite magnetization at the critical point. Following ref. 14 and using the finite-size scale argument for the magnetization and for the susceptibility per site, we can write $\beta = -\nu(D-2)$ and $\gamma = 2\nu(D-1)$. Since $1/\nu = 2\ln(1 + t_c^2)/\ln 2 = 0.747235...$, we end up with $\beta = 0.1617343...$ and $\gamma = -\ln(t_c)/\ln(1 + t_c^2) \cong 2.3530632...$, the former recovering the direct calculation [see Eq. (9)].

In summary, we found that the local magnetization of the pure ferromagnetic Ising model in the diamond hierarchical lattice has an intrinsic multifractal distribution at the critical point. An exact recurrence equation has been given for the local magnetization of each site for $T \leqslant T_c$. From this equation several results have been obtained, including the $f(\alpha)$ function describing the multifractality, distinct β-critical exponents for the local magnetization of the deepest sites of the DHL and for the average magnetization (per site), the fractal dimension of the ensemble of sites with finite magnetization at the critical point, and the critical exponent γ of the susceptibility. We remark that an infinite set of β exponents is expected to describe the critical behavior of the infinite classes of sites underlying the multifractal local magnetization profile of the DHL. The study of this infinite set of exponents is now in progress.

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